ON THE COHOMOLOGY OF THE LOSEV–MANIN MODULI SPACE

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Abstract. We determine the cohomology of the Losev–Manin moduli space $\overline{M}_{0,2|n}$ of pointed genus zero curves as a representation of the product of symmetric groups $\mathbb{S}_2 \times \mathbb{S}_n$.

Introduction

The Losev–Manin moduli space $\overline{M}_{0,2|n}$ was introduced in [6] and it parametrizes stable chains of projective lines with marked points $x_0 \neq x_\infty$ and $y_1, \ldots, y_n$, where the points $y_1, \ldots, y_n$ are allowed to collide, but not with x_0 nor x_∞, see Definition 1.1. In [6] this moduli space was denoted by $\overline{L}_n$, here we have adapted the notation used in [8]. There is a natural action of $\mathbb{S}_2 \times \mathbb{S}_n$ on $\overline{M}_{0,2|n}$ by permuting x_0, x_∞ and $y_1, \ldots, y_n$ respectively. This makes the cohomology $H^*(\overline{M}_{0,2|n}, \mathbb{Q})$ into a representation of $\mathbb{S}_2 \times \mathbb{S}_n$. The aim of this note is to determine the character of this representation.

The moduli space $\overline{M}_{0,2|n}$ can also be described as a moduli space of weighted pointed curves which were studied by Hassett [3, Section 6.4]. In this terminology it is the moduli space of genus 0 curves with 2 points of weight 1 and n points of weight $1/n$, and it would be written $\overline{M}_{0,\mathcal{A}}$ where $\mathcal{A} = (1, 1, \underbrace{1/n, \ldots, 1/n}_{n})$.

Another interesting aspect of the space $\overline{M}_{0,2|n}$ is that it has a structure of toric variety. It is proved in [6] that $\overline{M}_{0,2|n}$ is isomorphic to the smooth projective toric variety associated with the convex polytope called the permutahedron. This toric variety is obtained by an iterated blow-up of $\mathbb{P}^{n-1}$ formed by first blowing up n general points, then blowing up the strict transforms of the lines joining pairs among the original n points, and so on up to $(n-3)$-dimensional hyperplanes, see [4, §4.3]. With this perspective, the action of $\mathbb{S}_2 \times \mathbb{S}_n$ can be seen in the following way. The $\mathbb{S}_n$-action comes from permuting the n-points of the blow-up, and the action of $\mathbb{S}_2$ comes from the Cremona transform of $\mathbb{P}^{n-1}$ induced by the group inversion of the torus $(\mathbb{C}^*)^{n-1} : (t_1, \ldots, t_{n-1}) \mapsto (t_1^{-1}, \ldots, t_{n-1}^{-1})$.

Alternatively, we can view our moduli space $\overline{M}_{0,2|n}$ as the toric variety $X(A_{n-1})$ associated to the fan formed by Weyl chambers of the root system of type A_{n-1} ($n \geq 2$), see [1]. The cohomology of $X(A_{n-1})$ is a representation of the Weyl group $W(A_{n-1}) \cong \mathbb{S}_n$ and this representation was studied in [9, 2, 12, 5]. On the other hand, $X(A_{n-1})$ has another automorphism coming from that of the Dynkin diagram. This automorphism together with the action of the Weyl group corresponds precisely to the $\mathbb{S}_2 \times \mathbb{S}_n$-action on $\overline{M}_{0,2|n}$.

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The cohomology of the moduli space $\overline{M}_{0,2|n}$ has also been studied by mathematical physicists, since it corresponds to the solutions of the so-called commutativity equations. For this perspective we refer to [6, 10] and the references therein.

The outline of the paper is as follows. In Section 1 we define $\overline{M}_{0,2|n}$ and we state some known results on its cohomology. Our main result is Theorem 2.3 where we give a formula for the $\mathbb{S}_2 \times \mathbb{S}_n$-equivariant Poincaré-Serre polynomial of $\overline{M}_{0,2|n}$. The main theorem is formulated in Section 2 and it is proved in Section 3. In Section 4 we present a formula for the generating series of the $\mathbb{S}_2 \times \mathbb{S}_n$-equivariant Poincaré-Serre polynomial of $\overline{M}_{0,2|n}$. In Appendix A we then show that the result of Procesi in [9] on the $\mathbb{S}_n$-equivariant Poincaré-Serre polynomial is in agreement with our result. Finally in Appendix B we list the $\mathbb{S}_2 \times \mathbb{S}_n$-equivariant Poincaré-Serre polynomial of $\overline{M}_{0,2|n}$ for n up to 6.

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1. The moduli space $\overline{M}_{0,2|n}$

In this note, a curve means a compact and connected curve over $\mathbb{C}$ with at most nodal singularities and the genus of a curve is the arithmetic genus.

Definition 1.1. For $n \geq 1$, let $\overline{M}_{0,2|n}$ be the moduli space of genus 0 curves C with $n+2$ marked points $(x_0, x_\infty | y_1, \ldots, y_n)$ satisfying the following conditions:

(i) all the marked points are non-singular points of C,
(ii) x_0 and x_∞ are distinct,
(iii) $y_1, \ldots, y_n$ are distinct from x_0 and x_∞,
(iv) the components corresponding to the ends of the dual graph contain x_0 or x_∞,
(v) each component has at least three special (i.e. marked or singular) points.

Remark 1.2. In (iii) above, y_i and y_j are allowed to coincide. The conditions imply that the dual graph of C is linear and that each irreducible component must contain at least one marked point in $(y_1, \ldots, y_n)$. This means that C is a chain of projective lines of length at most n.

The moduli space $\overline{M}_{0,2|n}$ is a nonsingular projective variety of dimension $n-1$, see [6, Theorem 2.2]. It has an action of $\mathbb{S}_2 \times \mathbb{S}_n$ by permuting the marked points $(x_0, x_\infty | y_1, \ldots, y_n)$.

1.1. Cohomology of $\overline{M}_{0,2|n}$.

The cohomology ring $H^*(\overline{M}_{0,2|n}, \mathbb{Q})$ was studied in [6]. It is algebraic, i.e., all the odd cohomology groups are zero and $H^*(\overline{M}_{0,2|n}, \mathbb{Q})$ is isomorphic to the Chow ring $A^*(\overline{M}_{0,2|n}, \mathbb{Q})$, see [6, Theorem 2.7.1]. The Poincaré-Serre polynomials

$$E_{2|n}(q) = \sum_{i=0}^{n-1} \dim_{\mathbb{Q}} H^{2i}(\overline{M}_{0,2|n}, \mathbb{Q})\, q^i \in \mathbb{Z}[q]\,,$$

were also computed, see [6, Theorem 2.3].

The action of $\mathbb{S}_2 \times \mathbb{S}_n$ on $\overline{M}_{0,2|n}$ gives the cohomology $H^*(\overline{M}_{0,2|n}, \mathbb{Q})$ a structure of $\mathbb{S}_2 \times \mathbb{S}_n$ representation. In [9], Procesi computed the $\mathbb{S}_n$-equivariant Poincaré-Serre polynomial of the toric variety $X(A_{n-1})$ (which is isomorphic to $\overline{M}_{0,2|n}$), see Appendix A.

Throughout this note the coefficients of all cohomology groups will be $\mathbb{Q}$.

2. Statement of the result

2.1. **Partitions.** A partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ is a non-incresing sequence of non-negative integers which contains only finitely many non-zero λ_i's. The number $l(\lambda)$ of positive entries is called the *length* of λ. The number $|\lambda| := \sum_i \lambda_i$ is called the *weight* of λ. If $|\lambda| = n$ we say that λ is a partition of n. We denote by $\mathcal{P}(n)$ the set of partitions of n and by $\mathcal{P}$ the set of all partitions. A sequence

$$w \cdot \lambda = (\lambda_{w(1)}, \lambda_{w(2)}, \ldots), \quad w \in \mathbb{S}_{l(\lambda)} ,$$

obtained by permuting the non-zero elements of λ is called an ordered partition of n. The number c_λ of distinct ordered partitions obtained from λ is given by

$$c_\lambda = \frac{l(\lambda)!}{\#\mathrm{Aut}(\lambda)} ,$$

where $\mathrm{Aut}(\lambda)$ is the subgroup of $\mathbb{S}_{l(\lambda)}$ consisting of the permutations which preserve λ. Let $m_k(\lambda) := \#\{i \mid \lambda_i = k\}$, we then have

$$\#\mathrm{Aut}(\lambda) = \prod_{k\geq 1} (m_k(\lambda)!) .$$

With this notation a partition λ can also be written as $\lambda = [1^{m_1(\lambda)}\, 2^{m_2(\lambda)} \cdots]$. For $\lambda \in \mathcal{P}(n)$ and $\mu \in \mathcal{P}(m)$ we then define $\lambda + \mu \in \mathcal{P}(m+n)$ by $m_k(\lambda+\mu) := \#\{i \mid \lambda_i = k\} + \#\{i \mid \mu_i = k\}$.

2.2. **Symmetric functions.** For proofs of the statements in this section see for instance [7]. Let $\Lambda^y := \varprojlim \mathbb{Z}[y_1, \ldots, y_n]^{\mathbb{S}_n}$ be the ring of symmetric functions. Similarly we define $\Lambda^{x|y} := \Lambda^x \otimes \Lambda^y$. It is known that $\Lambda^y \otimes \mathbb{Q} = \mathbb{Q}[p_1^y, p_2^y, \ldots]$ where p_n^y are the power sums in the variable y. For $\lambda \in \mathcal{P}$, we set $p_\lambda^y := \prod_i p_{\lambda_i}^y$.

For a representation V of $\mathbb{S}_n$, we define

$$\mathrm{ch}_n^y(V) := \frac{1}{n!} \sum_{w \in \mathbb{S}_n} \mathrm{Tr}_V(w) p_{\rho(w)}^y \in \Lambda^y ,$$

where $\rho(w) \in \mathcal{P}(n)$ is the partition of n which represents the cycle type of $w \in \mathbb{S}_n$. Similarly we define, for a $\mathbb{S}_2 \times \mathbb{S}_n$ representation V,

$$\mathrm{ch}_{2|n}^{x|y}(V) := \frac{1}{2(n!)} \sum_{(v,w) \in \mathbb{S}_2 \times \mathbb{S}_n} \mathrm{Tr}_V\big((v,w)\big) p_{\rho(v)}^x\, p_{\rho(w)}^y \in \Lambda^{x|y} .$$

Recall that irreducible representations of $\mathbb{S}_n$ are indexed by $\mathcal{P}(n)$. For $\lambda \in \mathcal{P}(n)$, let V_λ be the irreducible representation corresponding to λ and define the Schur polynomial

$$s_\lambda^y := \mathrm{ch}_n^y(V_\lambda) \in \Lambda^y .$$

In the following we will use that, if V_i are representations of $\mathbb{S}_{n_i}$ for $1 \leq i \leq k$, then

$$\mathrm{ch}^y_{\sum_{i=1}^k n_i}\Big(\mathrm{Ind}^{\mathbb{S}_{\sum_{i=1}^k n_i}}_{\mathbb{S}_{n_1}\times\ldots\times\mathbb{S}_{n_k}}(V_1 \boxtimes \ldots \boxtimes V_k)\Big) = \prod_{i=1}^k \mathrm{ch}^y_{n_i}(V_i),$$

$$\mathrm{ch}^y_{n_1 n_2}\Big(\mathrm{Ind}^{\mathbb{S}_{n_1 n_2}}_{\mathbb{S}_{n_1}\sim\mathbb{S}_{n_2}}(V_1 \boxtimes \underbrace{V_2 \boxtimes \ldots \boxtimes V_2}_{n_1})\Big) = \mathrm{ch}^y_{n_1}(V_1) \circ \mathrm{ch}^y_{n_2}(V_2),$$

where $\sim$ denotes the wreath product, that is, $\mathbb{S}_{n_1} \sim \mathbb{S}_{n_2} := \mathbb{S}_{n_1} \ltimes (\mathbb{S}_{n_2})^{n_1}$ where $\mathbb{S}_{n_1}$ acts on $(\mathbb{S}_{n_2})^{n_1}$ by permutation, see [7, Appendix A, p. 158]. Plethysm is an operation $\circ : \Lambda^y \times \Lambda^y \to \Lambda^y$ which we will extend to an operation $\circ : \Lambda^y \times \Lambda^y[q] \to \Lambda^y[q]$ by putting $p^y_n \circ q = q^n$.

2.3. **The main theorem.**

Definition 2.1. The $\mathbb{S}_2 \times \mathbb{S}_n$-equivariant Poincaré-Serre polynomial of $\overline{M}_{0,2|n}$ is defined by

$$E_{\mathbb{S}_2\times\mathbb{S}_n}(q) := \sum_{i=0}^{n-1} \mathrm{ch}^{x|y}_{2|n}\left(H^{2i}(\overline{M}_{0,2|n})\right) q^i \in \Lambda^{x|y}[q]\ .$$

The usual Poincaré-Serre polynomial $E_{2|n}(q)$ is recovered from the equivariant one by

$$\frac{\partial^2}{\partial(p^x_1)^2}\frac{\partial^n}{\partial(p^y_1)^n} E_{\mathbb{S}_2\times\mathbb{S}_n}(q) = E_{2|n}(q)\,.$$

We will make some ad-hoc definitions in order to formulate an explicit formula for $E_{\mathbb{S}_2\times\mathbb{S}_n}(q)$. The proof will then furnish an explanation to these definitions.

Definition 2.2. First put $g^y_0 := 1$, then for any $n \geq 1$ and any (unordered) partition λ put

$$f^y_n := \sum_{i=0}^{n-1}(-1)^i s^y_{(n-i,1^i)} q^{n-1-i}, \quad F^y_\lambda := \prod_{j=1}^{l(\lambda)} f^y_{\lambda_j}, \quad g^y_n := \sum_{i=0}^{n-1} s^y_{(n-i,1^i)} q^{n-1-i}.$$

Theorem 2.3. *We then have*

$$E_{\mathbb{S}_2\times\mathbb{S}_n}(q) = \frac{1}{2}(p^x_1)^2 \sum_{\lambda\in\mathcal{P}(n)} c_\lambda F^y_\lambda + \frac{1}{2}p^x_2 \sum_{k=0}^{\lfloor n/2\rfloor} g^y_{n-2k} \sum_{\mu\in\mathcal{P}(k)} c_\mu \left(p^y_2 \circ F^y_\mu\right)\ . \tag{2.1}$$

Results for $1 \leq n \leq 6$ obtained from (2.1) are listed in Appendix B.

3. Proof of Theorem 2.3

3.1. **Stratification of $\overline{M}_{0,2|n}$.** For $k \geq 0$, we denote by $\Delta_{n,k}$ the closed subset of $\overline{M}_{0,2|n}$ consisting of curves with at least k nodes. Let $\Delta^*_{n,k} := \Delta_{n,k} \setminus \Delta_{n,k+1}$ be the open part of $\Delta_{n,k}$ which corresponds to curves with exactly k nodes. It is easy to see that $\Delta_{n,k} \neq \emptyset$ only for $0 \leq k \leq n-1$ and that $\Delta^*_{n,n-1} = \Delta_{n,n-1} = \{\mathrm{pt}\}$. Note that $\Delta^*_{n,k}$ is preserved by the $\mathbb{S}_2 \times \mathbb{S}_n$-action. Hence its cohomology $H^*(\Delta^*_{n,k})$ is a representation of $\mathbb{S}_2 \times \mathbb{S}_n$.

Definition 3.1. For an ordered partition λ of n with length $k+1$, let $\Delta^*_\lambda \subset \Delta^*_{n,k}$ correspond to all chains of projective lines of length $k+1$ such that precisely λ_i of the marked points $(y_1, \ldots, y_n)$ are on the ith component (where the component with the marked point x_0 is the 1st component and the one with x_∞ is the $(k+1)$th).

Note that Δ^*_λ is preserved by $\mathbb{S}_n$ (but not necessarily by $\mathbb{S}_2 \times \mathbb{S}_n$, see below) and hence $H^*(\Delta^*_\lambda)$ is a representation of $\mathbb{S}_n$.

Lemma 3.2. *(i)* $\Delta^*_{n,0} \cong (\mathbb{C}^*)^{n-1}$. *(ii)* $\Delta^*_\lambda \cong \prod_{i=1}^{l(\lambda)} \Delta^*_{\lambda_i,0}$.
(iii) We have a stratification

$$\Delta^*_{n,k} = \bigsqcup_{\lambda=(\lambda_1,\ldots,\lambda_{k+1})} \Delta^*_\lambda \ ,$$

where λ runs over all ordered partitions of n with length $k+1$.

Proof. (i) We have $\Delta^*_{n,0} \cong \left(\mathbb{P}^1 \setminus \{0,\infty\}\right)^n / \mathbb{C}^* \cong (\mathbb{C}^*)^n / \mathbb{C}^*$. (ii) Clear from the definition. (iii) This is found by considering the ways to distribute n marked points $(y_1,\ldots,y_n)$ on the chain of projective lines of length $k+1$ so that each irreducible component contains at least one of the points. □

It follows from Lemma 3.2 (ii) that Δ^*_λ and $\Delta^*_{\lambda'}$ are ($\mathbb{S}_n$-equivariantly) isomorphic when λ and λ' are different orderings of the same element in $\mathcal{P}(n)$.

3.2. **Cohomology of $\Delta^*_{n,0}$.** Since $\Delta^*_{n,0} \cong (\mathbb{C}^*)^{n-1}$, $H^i(\Delta^*_{n,0}) = 0$ for $i \geq n$, and moreover the mixed Hodge structure on $H_c^{2(n-1)-i}(\Delta^*_{n,0})$ is a pure Tate structure of weight $2(n-1-i)$, that is,

$$H_c^{2(n-1)-i}(\Delta^*_{n,0}) = \mathbb{Q}\big(-(n-1-i)\big)^{\oplus \binom{n-1}{i}} \ .$$

Lemma 3.3. *For $0 \leq i \leq n-1$, we have*

$$\mathrm{ch}_{2|n}^{x|y}\left(H^i(\Delta^*_{n,0})\right) = \begin{cases} s^x_{(2)}\, s^y_{(n-i,1^i)} & \text{if } i \text{ is even} \\ s^x_{(1^2)}\, s^y_{(n-i,1^i)} & \text{if } i \text{ is odd.} \end{cases}$$

Proof. Take an isomorphism $\Delta^*_{n,0} = (\mathbb{C}^*)^n / \mathbb{C}^* \to (\mathbb{C}^*)^{n-1}$ given by

$$(z_1 : z_2 : \cdots : z_{n-1} : z_n) \mapsto (\frac{z_1}{z_n}, \ldots, \frac{z_{n-1}}{z_n}) =: (y_1, \ldots, y_{n-1}) \ .$$

Then it is easy to see that $H^1(\Delta^*_{n,0}) = \oplus_{i=1}^{n-1} \mathbb{Q}[\frac{1}{2\pi\sqrt{-1}} \frac{dy_i}{y_i}]$ is the standard representation $s_{(n-1,1)}$ under the action of $\mathbb{S}_n$. The action of $\mathbb{S}_2$ is by interchanging 0 and ∞, that is by the isomorphism $t \mapsto 1/t$ of $\mathbb{P}^1$, which induces the action $(z_1 : \cdots : z_n) \mapsto (1/z_1 : \cdots : 1/z_n)$ on $\Delta^*_{n,0}$. This tells us that $(y_1, \ldots, y_{n-1}) \mapsto (1/y_1, \ldots, 1/y_{n-1})$ and since $\frac{d(1/y)}{1/y} = -\frac{dy}{y}$ we conclude that $H^1(\Delta^*_{n,0}) = V_{(1^2)} \boxtimes V_{(n-1,1)}$. Using once more that $\Delta^*_{n,0} \cong (\mathbb{C}^*)^{n-1}$ we get

$$H^k(\Delta^*_{n,0}) \cong \wedge^k H^1(\Delta^*_{n,0}) \cong \wedge^k (V_{(1^2)} \boxtimes V_{(n-1,1)}) \cong (\otimes^k V_{(1^2)}) \boxtimes V_{(n-k,1^k)} \ .$$

□

Corollary 3.4. *We have the equality*

$$\sum_{i=0}^{n-1} (-1)^i\, \mathrm{ch}_{2|n}^{x|y}\left(H_c^{2(n-1)-i}(\Delta^*_{n,0})\right) q^{n-1-i} = \frac{1}{2}(p_1^x)^2\, f_n^y + \frac{1}{2} p_2^x\, g_n^y \ .$$

Proof. By Poincaré duality, $H_c^{2(n-1)-i}(\Delta^*_{n,0}) \cong H^i(\Delta^*_{n,0})^\vee$, and since every irreducible representation of $\mathbb{S}_2 \times \mathbb{S}_n$ is defined over $\mathbb{Q}$, the dual representation is isomorphic to itself. The equality now follows from the lemma together with the relations $2s^x_{(2)} = (p^x_1)^2 + p^x_2$ and $2s^x_{(1^2)} = (p^x_1)^2 - p^x_2$. □

3.3. **Cohomology of Δ^*_λ.**

Corollary 3.5. *For any ordered partition λ of n with length $k+1$, $H_c^{2(n-k-1)-i}(\Delta^*_\lambda)$ is a pure Hodge structure of weight $2(n-k-1-i)$.*

Proof. This follows from Lemma 3.2 (ii) and the purity of the cohomology of $\Delta^*_{i,0}$. □

Corollary 3.6. *For any ordered partition λ of n with length $k+1$ we have*

$$\sum_{i=0}^{n-k-1} (-1)^i \operatorname{ch}^y_n \left(H_c^{2(n-k-1)-i}(\Delta^*_\lambda) \right) q^{n-k-1-i} = F^y_\lambda \ .$$

Proof. From Lemma 3.2 (ii) we know that $\Delta^*_\lambda \cong \prod_{i=1}^{k+1} \Delta^*_{\lambda_i,0}$, and on each $\Delta^*_{\lambda_i,0}$ we have an action of $\mathbb{S}_{\lambda_i}$. The action of $\mathbb{S}_n$ on $H^*_c(\Delta^*_\lambda)$ will thus be the induced action from $\mathbb{S}_{\lambda_1} \times \ldots \times \mathbb{S}_{\lambda_{k+1}}$ to $\mathbb{S}_n$. The result now follows from Corollary 3.4, forgetting the action of $\mathbb{S}_2$. □

3.4. **Proof of Theorem 2.3.** We have the following long exact sequence of cohomology with compact support:

$$\cdots \longrightarrow H_c^{i-1}(\Delta_{n,k+1}) \longrightarrow H_c^i(\Delta^*_{n,k}) \longrightarrow H_c^i(\Delta_{n,k}) \longrightarrow H_c^i(\Delta_{n,k+1}) \longrightarrow \cdots \ . \tag{3.1}$$

This is an exact sequence of both mixed Hodge structures and $\mathbb{S}_2 \times \mathbb{S}_n$-representations. Therefore, using the exact sequence (3.1) inductively (this is just the additivity of the Poincaré-Serre polynomial) we get

$$E_{\mathbb{S}_2 \times \mathbb{S}_n}(q) = \sum_{k=0}^{n-1} \left\{ \sum_{i=0}^{n-1} (-1)^i \operatorname{ch}^{x|y}_{2|n} \left(H_c^{2(n-1)-i}(\Delta^*_{n,k}) \right) q^{n-1-i} \right\} \ . \tag{3.2}$$

We will now find a formula for $\operatorname{ch}^{x|y}_{2|n}\big(H_c^{2(n-1)-i}(\Delta^*_{n,k})\big)$. Let us begin with a strata Δ^*_λ for an ordered partition λ of n with length $k+1$. The action of $\mathbb{S}_2$ will then send the strata given by λ to the one given by $\lambda' = (\lambda_{k+1}, \lambda_k, \ldots, \lambda_1)$. We will therefore divide into two cases.

Let us first assume that $\lambda \neq \lambda'$. Since the action of $\mathbb{S}_2$ interchanges the two components it will also interchange the factors of $H^i_c(\Delta^*_\lambda \sqcup \Delta^*_{\lambda'}) = H^i_c(\Delta^*_\lambda) \oplus H^i_c(\Delta^*_{\lambda'})$ and hence

$$\operatorname{ch}^{x|y}_{2|n}\big(H^i_c(\Delta^*_\lambda \sqcup \Delta^*_{\lambda'})\big) = (p^x_1)^2 \operatorname{ch}^y_n\big(H^i_c(\Delta^*_\lambda)\big) \ . \tag{3.3}$$

Let us now assume that $\lambda = \lambda'$. We can then decompose our space as $\Delta^*_\lambda = \Delta^*_1 \times \Delta^*_2 \times \Delta^*_3$ where, if $k+1 = 2m$,

$$\Delta^*_1 := \prod_{i=1}^{m} \Delta^*_{\lambda_i,0}, \quad \Delta^*_2 := \{\mathrm{pt}\}, \quad \Delta^*_3 := \prod_{i=m+1}^{2m} \Delta^*_{\lambda_i,0} \ ,$$

and, if $k+1=2m+1$,

$$\Delta_1^* := \prod_{i=1}^{m} \Delta^*_{\lambda_i,0}, \quad \Delta_2^* := \Delta^*_{\lambda_{m+1},0}, \quad \Delta_3^* := \prod_{i=m+2}^{2m+1} \Delta^*_{\lambda_i,0} \; .$$

Let us put $\alpha := \lambda_{m+1}$ if $k+1$ is odd and $\alpha := 1$ if $k+1$ is even, and in both cases $\beta := \sum_{i=1}^{m} \lambda_i$. The action of $\mathbb{S}_2$ interchanges the ($\mathbb{S}_\beta$-equivariantly) isomorphic components Δ_1^* and Δ_3^* and sends the space Δ_2^* to itself. Define the semidirect product $\mathbb{S}_2 \ltimes (\mathbb{S}_\beta \times \mathbb{S}_\alpha \times \mathbb{S}_\beta)$ where $\mathbb{S}_2$ acts as the identity on $\mathbb{S}_\alpha$ and permutes the factors $\mathbb{S}_\beta \times \mathbb{S}_\beta$ (i.e. as the wreath product). The group $\mathbb{S}_2 \ltimes (\mathbb{S}_\beta \times \mathbb{S}_\alpha \times \mathbb{S}_\beta)$ naturally embeds, by the map i say, in $\mathbb{S}_{2\beta+\alpha} = \mathbb{S}_n$. Let us then put $\mathbb{S}_2 \ltimes (\mathbb{S}_\beta \times \mathbb{S}_\alpha \times \mathbb{S}_\beta)$ in $\mathbb{S}_2 \times \mathbb{S}_n$ by $(\tau,\sigma) \mapsto (\tau, i(\tau,\sigma))$, where $\tau \in \mathbb{S}_2$ and $\sigma \in \mathbb{S}_\beta \times \mathbb{S}_\alpha \times \mathbb{S}_\beta$. The action of $\mathbb{S}_2 \times \mathbb{S}_n$ on Δ^*_λ will then be the induced action from $\mathbb{S}_2 \ltimes (\mathbb{S}_\beta \times \mathbb{S}_\alpha \times \mathbb{S}_\beta)$ acting naturally on $\Delta_1^* \times \Delta_2^* \times \Delta_3^*$. Using Corollary 3.4 we conclude that

$$\text{(3.4)} \quad \mathrm{ch}^{x|y}_{2|n}\big(H^i_c(\Delta^*_\lambda)\big) = \frac{1}{2} p^x_{(1^2)}\, f^y_\alpha \left(p^y_{(1^2)} \circ \mathrm{ch}^y_\beta\big(H^i_c(\Delta_1^*)\big)\right) + \frac{1}{2} p^x_{(2)}\, g^y_\alpha \left(p^y_{(2)} \circ \mathrm{ch}^y_\beta\big(H^i_c(\Delta_1^*)\big)\right) .$$

Applying formula (3.3) and formula (3.4) (and using Lemma 3.2 (iii) and Corollary 3.6) to equation (3.2), gives equation (2.1).

4. Generating series

4.1. **Generating series of $E_{\mathbb{S}_2\times\mathbb{S}_n}(q)$.** For any sequence of polynomials h_n we have the formal identity,

$$\text{(4.1)} \qquad 1 + \sum_{n=1}^{\infty}\Big(\sum_{\lambda\in\mathcal{P}(n)} c_\lambda \prod_{j=1}^{l(\lambda)} h_{\lambda_j}\Big) = 1 + \sum_{r=1}^{\infty}\Big(\sum_{n=1}^{\infty} h_n\Big)^r = \Big(1 - \sum_{n=1}^{\infty} h_n\Big)^{-1}.$$

The following proposition follows directly from (4.1) and Theorem 2.3.

Proposition 4.1. *The generating series of $E_{\mathbb{S}_2\times\mathbb{S}_n}(q)$ is determined by,*

$$\text{(4.2)} \; 1 + \sum_{n=1}^{\infty} E_{\mathbb{S}_2\times\mathbb{S}_n}(q) = \frac{1}{2}(p_1^x)^2 \left(1 - \sum_{n=1}^{\infty} f^y_n\right)^{-1} + \frac{1}{2} p_2^x \left(1 + \sum_{n=1}^{\infty} g^y_n\right)\left(1 - \sum_{n=1}^{\infty} (p_2^y \circ f^y_n)\right)^{-1}$$

Remark 4.2. Consider the moduli space M defined as in Definition 1.1 but with the additional demand that $y_1, \ldots, y_n$ are distinct from each other. From Carel Faber we learnt the following formula, which is very similar to (4.2), for the generating series of the $\mathbb{S}_2 \times \mathbb{S}_n$-equivariant Poincaré-Serre polynomial of M. Carel Faber obtained the formula as a direct consequence of an equality he learned from Ezra Getzler. These results have not been published.

Let h^y_{n+2} be the $\mathbb{S}_{n+2}$-equivariant Poincaré-Serre polynomial of $M_{0,n+2}$, the moduli space of genus 0 curves with $n+2$ marked distinct points. The $\mathbb{S}_2 \times \mathbb{S}_n$-equivariant Poincaré-Serre polynomial of the open part of M (defined using the compactly supported Euler-characteristic) consisting of irreducible curves will then equal

$$\frac{1}{2}(p_1^x)^2\, \tilde f^y_n + \frac{1}{2} p_2^x\, \tilde g^y_n = \frac{1}{2}(p_1^x)^2 \left(\frac{\partial^2 h^y_{n+2}}{\partial (p^y_1)^2}\right) + \frac{1}{2} p_2^x \left(2\, \frac{\partial h^y_{n+2}}{\partial p_2}\right).$$

From the proof of Theorem 2.3 we see that replacing f_n^y by $\tilde{f}_n^y$ (and g_n^y by $\tilde{g}_n^y$) in equation (4.2) gives the $\mathbb{S}_2 \times \mathbb{S}_n$-equivariant Poincaré-Serre polynomial of M.

Remark 4.3. The polynomials f_n^y and g_n^y can be formulated in terms of $P_\lambda^y(q) \in \Lambda^y[q]$, the Hall–Littlewood symmetric function associated to $\lambda \in \mathcal{P}$ (cf. [7, III-2]). This function is defined as the limit of the following symmetric polynomial:

$$P_\lambda(y_1, \dots, y_k; q) = \sum_{w \in \mathbb{S}_k/\mathbb{S}_k^\lambda} w \left(y_1^{\lambda_1} \cdots y_k^{\lambda_k} \prod_{\lambda_i > \lambda_j} \frac{y_i - q y_j}{y_i - y_j} \right) ,$$

where $\mathbb{S}_k^\lambda$ is the stabilizer subgroup of λ in $\mathbb{S}_k$ and $l(\lambda) \leq k$ is assumed. In the special case $\lambda = (n)$, where $n \geq 1$, the following formula is known (cf. [7, p. 214]):

$$P_{(n)}^y(q) = \sum_{r=0}^{n-1} (-q)^r s^y_{(n-r,1^r)} , \tag{4.3}$$

hence $f_n^y = q^{n-1} P^y_{(n)}(q^{-1})$ and $g_n^y = q^{n-1} P^y_{(n)}(-q^{-1})$.

4.2. **Generating series of $E_{\mathbb{S}_n}(q)$.** The $\mathbb{S}_n$-equivariant Poincaré-Serre polynomial of $\overline{M}_{0,2|n}$ equals

$$E_{\mathbb{S}_n}(q) := \sum_{i=0}^{n-1} \mathrm{ch}_n^y \left(H^{2i}(\overline{M}_{0,2|n}) \right) q^i = \frac{\partial^2}{\partial (p_1^x)^2} E_{\mathbb{S}_2 \times \mathbb{S}_n}(q) \in \Lambda^y[q] ,$$

and so

$$1 + \sum_{n=1}^{\infty} E_{\mathbb{S}_n}(q) = \Big(1 - \sum_{n=1}^{\infty} f_n^y \Big)^{-1} . \tag{4.4}$$

Corollary 3.4 then tells us that the generating series of $E_{\mathbb{S}_n}(q)$ is the multiplicative inverse of the generating series (in compactly supported cohomology) of $\Delta^*_{n,0}$, which is the open part of $\overline{M}_{0,2|n}$ consisting of irreducible curves.

If we set $q = 1$, the Hall–Littlewood function $P^y_{(n)}(q^{-1})$ becomes the nth power sum p_n^y and formula (4.4) takes a very simple form. Let $e_{\mathbb{S}_n} := E_{\mathbb{S}_n}(1) \in \Lambda^y$, be the $\mathbb{S}_n$-equivariant Euler characteristic of $\overline{M}_{0,2|n}$. We then have

$$1 + \sum_{n=1}^{\infty} e_{\mathbb{S}_n} z^n = \left(1 - \sum_{n=1}^{\infty} p_n^y z^n \right)^{-1} .$$

Appendix A. Consistency with Procesi's result

A.1. **Procesi's recursive formula.** In [9], Procesi obtained the following recursive relation among $E_{\mathbb{S}_n}(q)$ with respect to n.

Theorem A.1 (Procesi)**.** *The $E_{\mathbb{S}_n}(q)$ satisfy*

$$E_{\mathbb{S}_{n+1}}(q) = s^y_{(n+1)} \sum_{i=0}^{n} q^i + \sum_{i=0}^{n-2} s^y_{(n-i)} \, E_{\mathbb{S}_{i+1}}(q) \left(\sum_{k=1}^{n-i-1} q^k \right) .$$

As a corollary, we have the following formula which is obtained in [2, 11, 12].

Corollary A.2. *We have*

$$1+\sum_{n=1}^{\infty} E_{\mathbb{S}_n}(q)t^n = \frac{(1-q)H(t)}{H(qt)-qH(t)} \ ,$$

where $H(t)=\sum_{r\geq 1} h_r t^r$ *is the generating function of the complete symmetric functions in the variable* y.

A.2. **Equivalence.** The following proposition shows the equivalence between our result and Procesi's by comparing Equation (4.4) and Equation (4.3) to Corollary A.2.

Proposition A.3. *We have*

$$\frac{(1-q)H(t)}{H(qt)-qH(t)} = \left\{1-\sum_{r=1}^{\infty} q^{-1}P^y_{(r)}(q^{-1})(qt)^r\right\}^{-1} \ .$$

Proof. As in [7, pp. 209–210], we have

$$\frac{H(qt)}{H(t)} = \prod_{i\geq 1}\frac{1-ty_i}{1-qty_i} = 1+(1-q^{-1})\sum_{i=1}^{n}\frac{y_iqt}{1-y_iqt}\prod_{j:j\neq i}\frac{y_i-q^{-1}y_j}{y_i-y_j} =$$

$$= 1+(1-q^{-1})\sum_{r=1}^{\infty}P^y_{(r)}(q^{-1})(qt)^r \ .$$

An easy manipulation of this formula gives the wanted equality. ∎

Appendix B. $E_{\mathbb{S}_2\times\mathbb{S}_n}(q)$ for n up to 6

n	$E_{\mathbb{S}_2\times\mathbb{S}_n}(q)$
1	$s^x_{(2)}s^y_{(1)}$
2	$(q+1)s^x_{(2)}s^y_{(2)}$
3	$s^x_{(2)}\Big((q^2+q+1)s^y_{(3)}+q\,s^y_{(2,1)}\Big)+q\,s^x_{(1^2)}s^y_{(3)}$
4	$s^x_{(2)}\Big((q^3+2q^2+2q+1)s^y_{(4)}+(q^2+q)s^y_{(3,1)}+(q^2+q)s^y_{(2^2)}\Big)$ $+s^x_{(1^2)}\Big((q^2+q)s^y_{(4)}+(q^2+q)s^y_{(3,1)}\Big)$
5	$s^x_{(2)}\Big((q^4+2q^3+4q^2+2q+1)s^y_{(5)}+(2q^3+3q^2+2q)s^y_{(4,1)}$ $+(q^3+3q^2+q)s^y_{(3,2)}+q^2\,s^y_{(2^2,1)}\Big)$ $+s^x_{(1^2)}\Big((2q^3+2q^2+2q)s^y_{(5)}+(q^3+3q^2+q)s^y_{(4,1)}+(q^3+2q^2+q)s^y_{(3,2)}+q^2\,s^y_{(3,1^2)}\Big)$
6	$s^x_{(2)}\Big((q^5+3q^4+6q^3+6q^2+3q+1)s^y_{(6)}+(2q^4+6q^3+6q^2+2q)s^y_{(5,1)}$ $+(2q^4+7q^3+7q^2+2q)s^y_{(4,2)}+(q^3+q^2)s^y_{(4,1^2)}+(2q^3+2q^2)s^y_{(3^2)}$ $+(2q^3+2q^2)s^y_{(3,2,1)}+(q^3+q^2)s^y_{(2^3)}\Big)$ $+s^x_{(1^2)}\Big((2q^4+4q^3+4q^2+2q)s^y_{(6)}+(2q^4+6q^3+6q^2+2q)s^y_{(5,1)}$ $+(q^4+5q^3+5q^2+q)s^y_{(4,2)}+(2q^3+2q^2)s^y_{(4,1^2)}$ $+(q^4+3q^3+3q^2+q)s^y_{(3^2)}+(2q^3+2q^2)s^y_{(3,2,1)}\Big)$

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